Econ 262A: Problem Set 3
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Question 1. Instrumental Variables and Marginal Treatment Effects Simulation

We're going to revisit the problem from last time, but with a continuous instrument. Suppose you have the following relationship in your data:

$$X_i \sim N(0,1)$$

$$Z_i \sim N(0,1)$$

$$\begin{pmatrix} \alpha_i \\ u_i \\ \varepsilon_i \end{pmatrix} \sim N\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & .25 & .25 \\ .25 & 1 & .25 \\ .25 & .25 & 1 \end{pmatrix} \right)$$

$$T_i = \mathbf{1}\left\{-.5 + .25X_i + Z_i + u_i \geq 0\right\}$$

$$Y_i = .5X_i + \alpha_i T_i + \varepsilon_i$$

As an example, imagine that we are studying the returns to college: Y is individual earnings, T is college or not, X is a personal characteristic, and Z is proximity to college. We take proximity to be broadly defined, as it can take negative and positive values.

The propensity score p_i is an individual's probability of being treated if they have values X and Z for observables and instrument, respectively,

$$p = E(T \mid X, Z) = \Phi(-.5 + .25X_i + Z_i).$$

Remember that you can rewrite the treatment choice as the condition

$$p_i = \Phi(-.5 + .25X_i + Z_i) > v_i,$$

where $v_i = \Phi(-u_i) \sim U[0,1]$, which we call the individual's resistance to treatment.

Part (a)

Constructing an individual's conditional expectation of their outcome as a function of X and p, $E(Y \mid X, p)$, show that the following two equalities hold:

$$\frac{\partial E(Y \mid X, p)}{\partial p} = E(\alpha_i \mid v_i = p_i) = 1 - 0.25 \times \Phi^{-1}(p)$$

Hint: You can use the Carneiro, Heckman, and Vytlacil (2011) derivation of the MTE to show this.

This derivation tells you that the derivative of the conditional expectation with respect to the propensity score at a certain value p should equal to the marginal treatment effect of individuals with resistance to treatment at percentile p. Note also that the marginal treatment effects are not a function of X, which we will exploit.

Since $\varepsilon_i \perp X_i, Z_i$, we have

$$\begin{aligned} E[Y_i|X_i,p] &= 0.5X_i + E[\alpha_i \mathbb{I}[v_i \le p]] \\ &= 0.5X_i + \int_0^p E(\alpha_i|v_i = s) f_v(s) ds \\ &= 0.5X_i + \int_0^p E(\alpha_i|v_i = s) ds, \end{aligned}$$

where the first line is from the treatment condition, the second comes from law of total expectation, and third comes from the fact that $v_i \sim U[0,1]$, so $f_v \equiv 1$. Differentiating w.r.t. p yields the first equality:

$$\frac{\partial E(Y \mid X, p)}{\partial p} = E(\alpha_i \mid v_i = p).$$

Computing the RHS, we have that the event $\{v_i = p\}$ equals the event $\{u_i = -\Phi^{-1}(p)\}$. Since α_i, u_i are jointly normal, we have that

$$E(\alpha_i|u_i) = E(\alpha_i) + \frac{\text{Cov}(\alpha_i, u_i)}{\text{Var}(u_i)}(u_i - E(u_i)) = 1 + 0.25(-\Phi^{-1}(p)),$$

proving the second equality. □

Since α_i, u_i are positively correlated, people with higher u_i (lower v_i resistance to treatment) have higher treatment effects (positive selection on gains). Increasing p means you are adding those on the margin with higher resistance to treatment, which corresponds to the negative term appended to the MTE.

Part (b)

Simulate 100,000 (NOTE: 10X from last time) observations from the above data-generating process. Plot the propensity score across individuals, and plot it separately for individuals with $X > 0$ and $X < 0$. Do we have full support unconditional on X? Do we have full support conditional on $X > 0$ and $X < 0$?

Propensity score across individuals is in Figure 1. Across $X > 0$ and $X < 0$ is in Figure 2. Since $Z_i \sim N(0,1)$ has full support on $\mathbb{R}$, the $-.5 + .25X_i + Z_i$ also has full support on $\mathbb{R}$, both unconditionally and conditional on X_i. Thus $p_i = \Phi(-.5 + .25X_i + Z_i)$ has support on $(0,1)$, at least in population. The finite, but large sample checks this.

Part (c)

Run a first-stage probit regression of T on X and Z, and save the predicted value $p = E(T \mid X, Z)$. Then run a regression of Y on X and a quadratic polynomial of p. Report the coefficients for both of these regressions in a table.

The regression outputs are shown in Table 1.

Using the regression on Y, how could we test whether treatment effects differ by resistance to treatment? Perform this test and report what you conclude.

Treatment effects differ by resistance to treatment if $MTE(p) = \frac{\partial E(Y|X,p)}{\partial p}$ is not constant in p, or $E(Y|X,p)$ is not linear in p. Since our regression included a quadratic term in p, call the coefficient on this term β_2. Our null hypothesis is $H_0 \colon \beta_2 = 0$, implying no difference in treatment

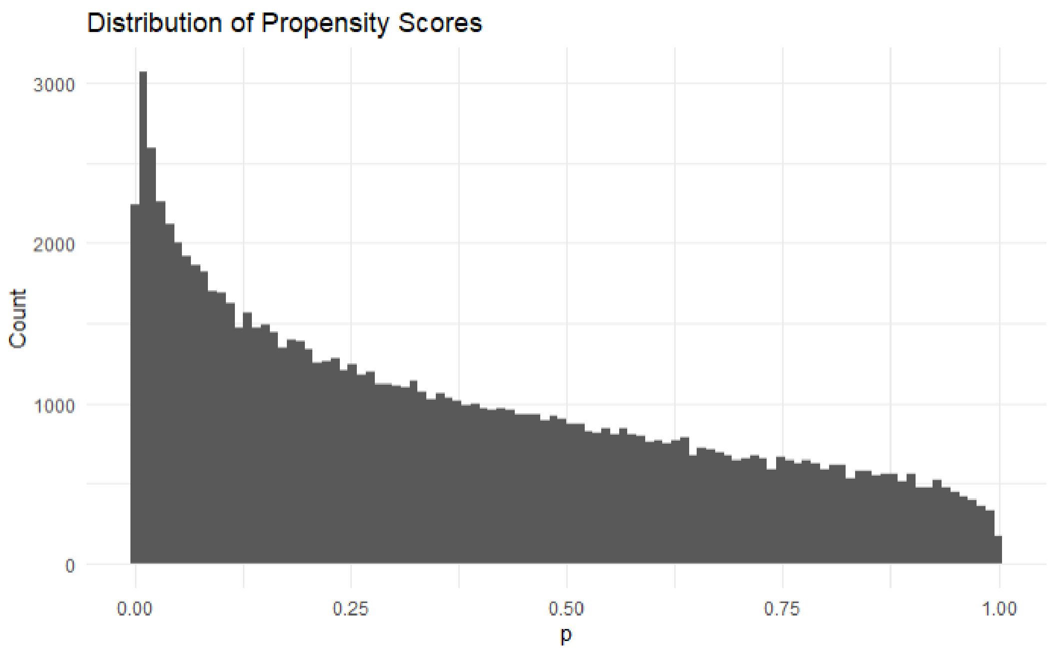

Figure 1: Propensity Score Across Individuals

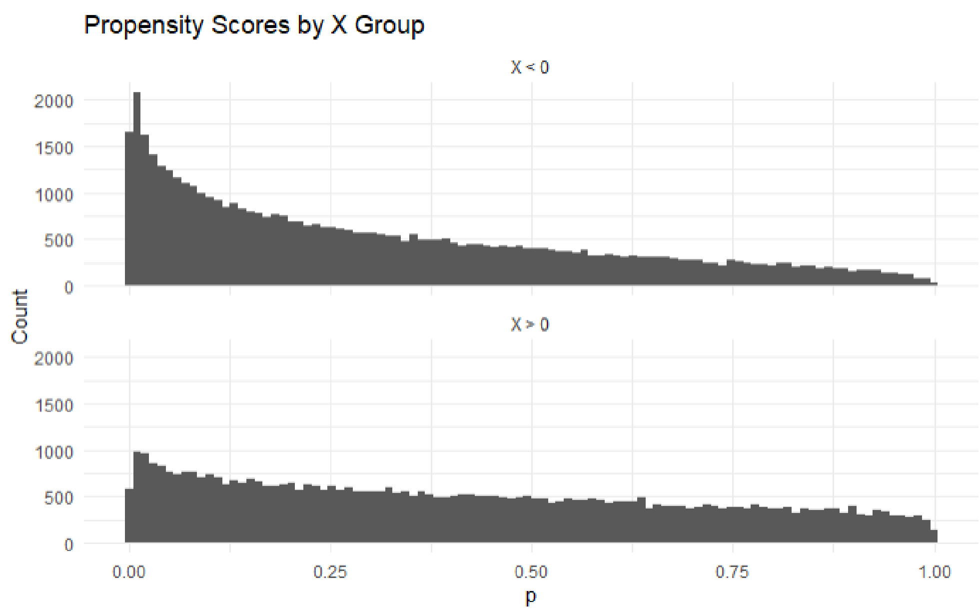

Figure 2: Propensity Scores by $X > 0, X < 0$

Table 1: First-stage and outcome regressions

	First-stage probit	Outcome regression
(Intercept)	−0.499***	−0.006
	(0.005)	(0.010)
X	0.255***	0.501***
	(0.005)	(0.005)
Z	0.998***	
	(0.006)	
p_hat		1.421***
		(0.056)
I(p_hat^2)		−0.409***
		(0.061)
Num.Obs.	100 000	100 000
R2		0.176

+ p <0.1, * p <0.05, ** p <0.01, *** p <0.001

effects by resistance to treatment, against the two-sided alternative $H_1\colon \beta_2 \neq 0$. From the output of the regression, we can reject the null with extremely high confidence and claim there differences by treatment resistance.

What is your interpretation of the coefficient on p? What is your interpretation of the coefficient of p^2? Do we have selection on gains, reverse selection on gains, or no selection on gains?

Letting the coefficient on p be β_1, we have that $MTE(p) = \beta_1 + 2\beta_2 p$. Thus, β_1 is the approximate marginal treatment effect at a hypothetical zero propensity score $p = 0$, and β_2 is the slope of the MTE, revealing if MTE changes as resistance to treatment changes. Since $\beta_2 < 0$, we have that MTE is decreasing in p, so higher-resistance individuals have lower treatment effects, and we have positive selection on gains.

What parameter in the simulation is determining whether we have positive or negative selection on gains? How could we change it to get the reverse selection pattern?

The key parameter is the covariance between α_i and u_i, i.e. the individual treatment effect and the idiosyncratic treatment-choice effect. If these have positive covariance, as we do in the simulation, lower resistance individuals have higher treatment effects, and we have positive Roy selection. If this is negative, then the opposite occurs and we have negative Roy selection, where the MTE curve slopes upwards.

Part (d)

Here we are going to use the `mtefe` command in Stata, from Martin Andresen, "Exploring Marginal Treatment Effects: Flexible estimation using Stata."

Run the `mtefe` command, where T is endogenous and Z is the instrument. Let the model be the default normal model. Provide the MTE figure and the table.

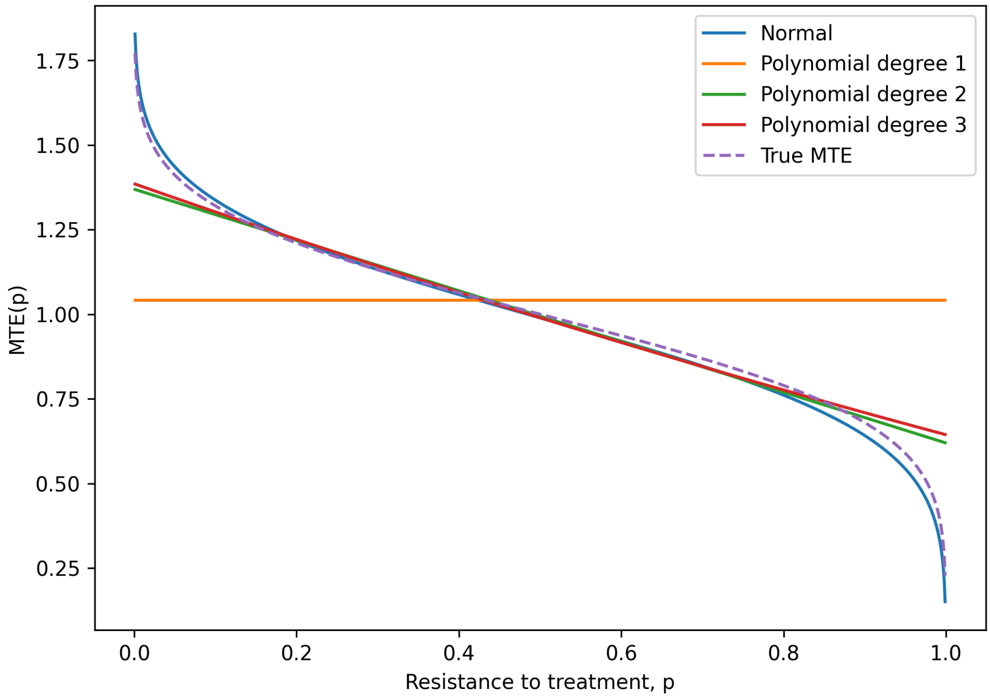

Figure 3: MTE figure, with normal model, and polynomial models, deg = 1,2,3

What is the intuition of the shape of the MTE figure? What is the intuition of the values for ATE, ATT, and ATUT?

Rerun the `mtefe` command using the normal model and polynomial models of degree 1, 2, and 3. Plot the MTE figures over each other (without standard errors). The MTE values are stored in `e(mte)`.

The `mtefe` output (run in Python since I do not have Stata) is given in Figure 3.

The table is given below.

Model	ATE	ATT	ATU
Normal	0.987	1.178	0.877
Polynomial degree 1	1.039	1.038	1.040
Polynomial degree 2	0.992	1.148	0.903
Polynomial degree 3	0.996	1.151	0.907
True MTE	0.998	1.174	0.897

The MTE curve is downward-sloping, i.e. individuals with low resistance to treatment have higher treatment effects, giving positive selection on gains. This is consistent with parts (a) and (c). The estimated treatment effects also showcase this. For all models except for the degree 1 model (which necessarily has constant MTE and doesn't capture selection on gains), ATT is larger than the ATE, while the ATU is smaller than the ATE. This occurs because treated individuals are drawn from the low-resistance part of the population, where the MTE is high, and vice versa for the untreated individuals.

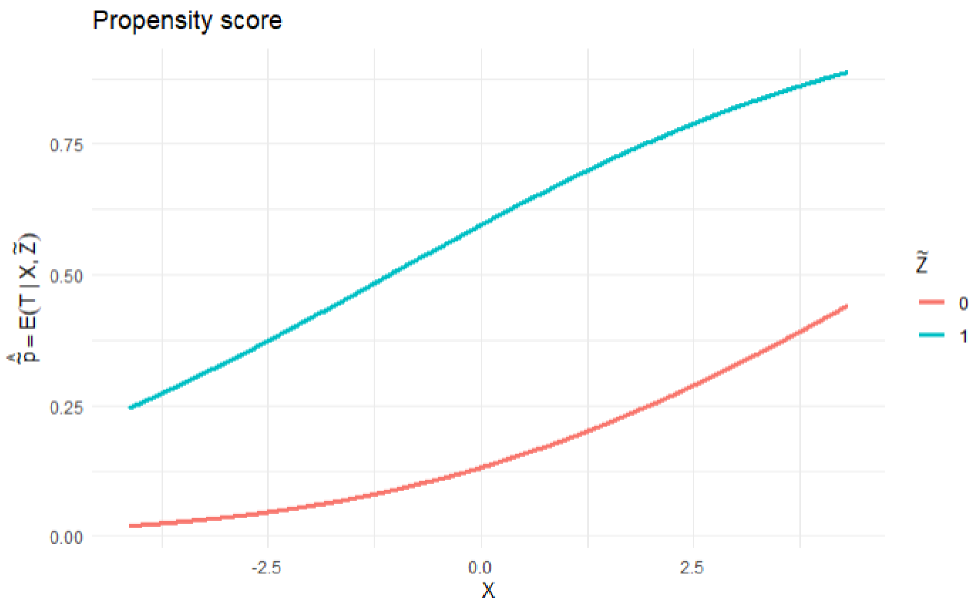

Figure 4: Propensity scores using binary instrument $\widetilde{Z}$

Part (e)

Now, construct a new instrument $\tilde{Z}$ that is a binary version of the true instrument Z:

$$\tilde{Z} = \mathbf{1}(Z \geq 0)$$

Is $\tilde{Z}$ a valid instrument for T? Plot the propensity score $\tilde{p} = E(T \mid X, \tilde{Z})$. Do we still have full support?

The plot of propensity score on X, partitioned by $\widetilde{Z}$ is shown in Figure 4. Notably, while $\widetilde{Z}$ is still a valid instrument since it is derived deterministically from Z, it is no longer a continuous instrument and only takes on values in $\{0, 1\}$. As is shown through the propensity scores, they do not comprise the full interval of $(0, 1)$ for each X as before, which is necessary for full nonparametric MTE identification.

Run the `mtefe` command using $\tilde{Z}$ instead of Z. Provide the output of the MTE figure and the table. How do you interpret these results in comparison with using `mtefe` with Z? What role does additive separability in the MTE play here?

The figure with the normal model is given in Figure 5.

The treatment effect outputs are given below.

Model	ATE	ATT	ATU
Normal	0.973	1.191	0.847
True MTE	0.998	1.198	0.883

Using the binary instrument $\widetilde{Z}$, the normal model is downward sloping and close to the true MTE. The treatment effects again display the $ATT > ATE > ATU$ trend, revealing positive selection on gains. This is not a nonparametric identification of the full MTE, however, since again,

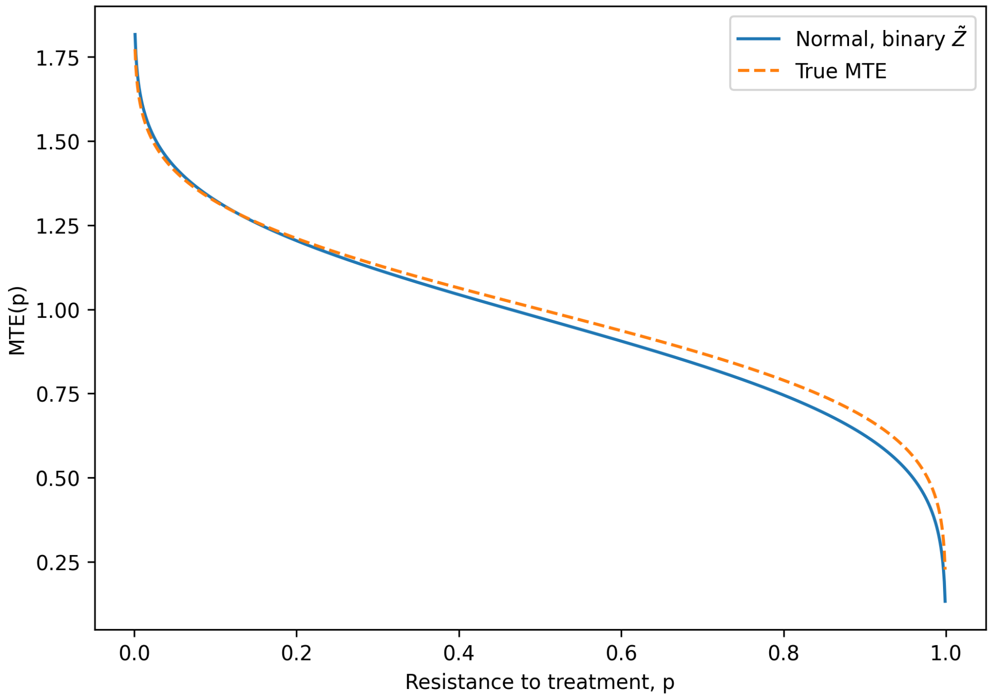

Figure 5: `mtefe` output

we only have two propensity score values for each X. Since the data-generating process is normal and additively separable, even though $\widetilde{Z}$ doesn't provide full support, the parametric normal model can extrapolate the MTE curve to propensity scores not generated by the binary instrument.

Part (f)

Conduct the Brinch, Mogstad, and Wiswall (2017) approach to estimating a linear MTE with a binary instrument. Specifically, apply Example 1 from Section 3.1 from the text.

Plot what your estimate of the linear MTE is, and plot it against the value of the normal MTE from Part (d). How do these MTE estimates compare?

Under the linear MTE restriction,

$$E(Y \mid D = 1, X, p) = \mu_1 + \beta_1 X + \frac{1}{2}\alpha_1(p - 1)$$

and

$$E(Y \mid D = 0, X, p) = \mu_0 + \beta_0 X + \frac{1}{2}\alpha_0 p.$$

Estimating these two equations separately and using p as the probability an individual gets treated (not dependent on X), we get the BMW linear MTE estimate:

$$MTE(p) = \mu_1 - \mu_0 + (\beta_1 - \beta_0)\bar{X} + (\alpha_1 - \alpha_0)(p - .5).$$

The estimated models are given in Figure 6. The BMW linear MTE captures the main downward-sloping/positive selection on gains pattern. However, it doesn't capture nonlinear patterns, especially near the endpoints $p = 0$ and $p = 1$. This is expected because the true MTE in this simulation

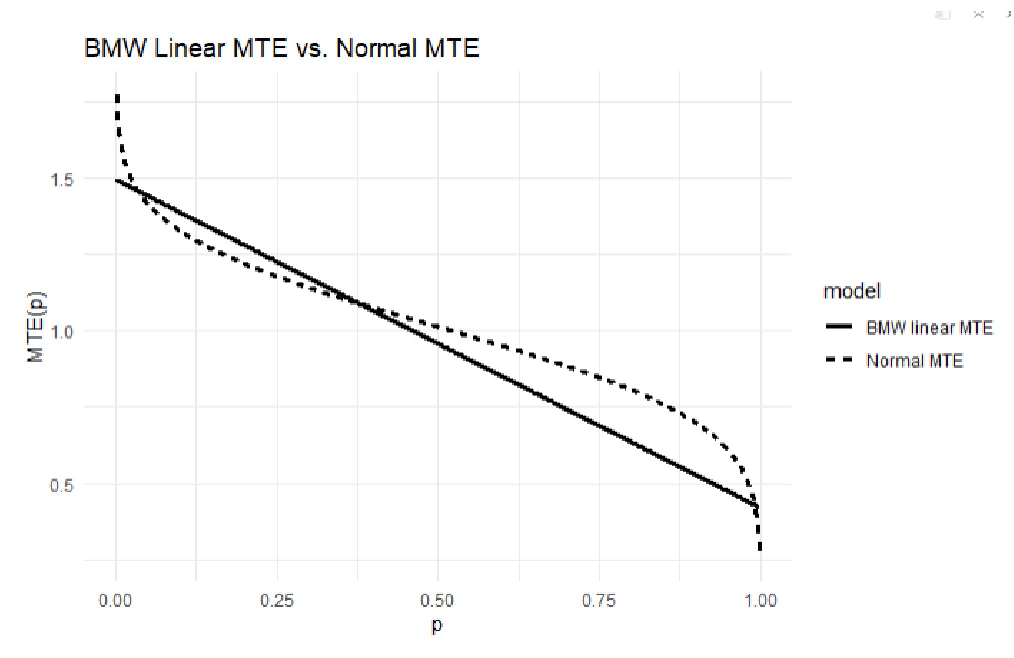

Figure 6: Linear MTE vs. Normal MTE

is

$$MTE(p) = 1 - 0.25\Phi^{-1}(p),$$

which is nonlinear in p.

Question 2. Counterfactual-Specific Treatment Effects

Simulate a data set with 1,000,000 observations (in your choice of program), and construct the following variables:

$$Z_{1i} = \begin{cases} 0 & \text{with probability } 0.5 \\ 1 & \text{with probability } 0.5 \end{cases}$$

$$Z_{2i} = \begin{cases} 0 & \text{with probability } 0.5 \\ 1 & \text{with probability } 0.5 \end{cases}$$

$$\begin{pmatrix} \alpha_i^1 \\ \alpha_i^2 \\ u_i^1 \\ u_i^2 \\ \varepsilon_i \end{pmatrix} \sim N\left(\begin{pmatrix} 0.5 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0.25 & -0.75 & 0.25 & 0.25 \\ 0.25 & 1 & 0.25 & 0.75 & 0.25 \\ -0.75 & 0.25 & 1 & 0.25 & 0.25 \\ 0.25 & 0.75 & 0.25 & 1 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 & 1 \end{pmatrix} \right)$$

$$U_{1i} = -.5 + Z_{1i} + u_i^1$$

$$U_{2i} = -.5 + Z_{2i} + u_i^2$$

$$T_{0i} = U_{1i} < 0 \ \& \ U_{2i} < 0$$

$$T_{1i} = U_{1i} \geq U_{2i} \ \& \ U_{1i} \geq 0$$

$$T_{2i} = U_{2i} > U_{1i} \ \& \ U_{2i} \geq 0$$
$$Y_i = \alpha_i^1 T_{1i} + \alpha_i^2 T_{2i} + \varepsilon_i$$

As an example, imagine that we are studying the returns to community college and four-year college: Y is individual earnings, T_{1i} is community college or not, T_{2i} is four-year college or not, and Z_{1i} and Z_{2i} are binary instruments for each. The individual chooses whichever options gives the highest utility U_{ki}, and the utility of no college is normalized to 0.

We will denote by Y_{i0}, Y_{i1}, and Y_{i2} the counterfactual outcomes for individuals with no college, community college, and four-year college, respectively. $T_i(z_1, z_2)$ is a function that inputs values for each instrument and outputs the treatment choice, $T \in \{0, 1, 2\}$.

We call someone a "0-1 complier at z_2" if $T(0, z_2) = 0$, $T(1, z_2) = 1$, and someone is a "0-1 complier" if $T(0, Z_2) = 0$, $T(1, Z_2) = 1$ for their realized value Z_2. We have analogous definitions for 2-1 compliers, 0-2 compliers, and 1-2 compliers.

Part (a)

Using the fact that you know U_{1i} and U_{2i} in your simulated data, calculate the treatment effects

$$E(Y_{i1} - Y_{i0} \mid \text{0-1 complier})$$

and

$$E(Y_{i2} - Y_{i0} \mid \text{0-2 complier}).$$

In other words, identify the people whose treatment switches from 0 to 1 when Z_1 switches from 0 to 1 (at their observed level of Z_2) and calculate their average treatment effect of this switch, and similarly for 0 to 2 compliers.

A 0-1 complier is an individual who has

$$T_i(0, Z_{2i}) = 0 \quad \text{and} \quad T_i(1, Z_{2i}) = 1,$$

and we measure the treatment effect of community college: $Y_{i1} - Y_{i0} = \alpha_i^1$.

A 0-2 complier is an individual who has

$$T_i(Z_{1i}, 0) = 0 \quad \text{and} \quad T_i(Z_{1i}, 1) = 2,$$

and we measure the treatment effect of four-year college: $Y_{i2} - Y_{i0} = \alpha_i^2$.

The estimates give LATEs $E(\alpha_i^1 | \text{0-1 complier}) = 0.198$ and $E(\alpha_i^2 | \text{0-2 complier}) = 0.944$.

Part (b)

Run a standard 2SLS regression where you regress Y on T_{1i} and T_{2i} and instrument using Z_{1i} and Z_{2i}. How do the estimated treatment effects compare to the complier treatment effects from Part (a)?

If they differ significantly, what is the explanation?

The estimates are given in Table 2. The treatment effect of T_1 is smaller and barely significant, while the treatment effect of T_2 is slightly larger than those computed in part (a). In a multinomial treatment setting, instruments can move individuals across multiple margins. For example, changing Z_1 can move individuals from T_0 to T_1, but also move some individuals from T_2 to T_1, mixing the

	2SLS
T_1	0.016
	(0.011)
T_2	1.053***
	(0.011)
Num.Obs.	1 000 000
R2	0.251

+ p <0.1, * p <0.05, ** p <0.01, *** p <0.001

Table 2: 2SLS Estimates

effects for $0-1$ compliers and $2-1$ compliers. The latter margin involves the treatment effect $Y_1 - Y_2$, which biases the estimate downwards. Similarly, the coefficient on T_2 mixes effects for $0-2$ compliers and $1-2$ compliers.

Part (c)

Let's now think about how to use the Mountjoy (2019) approach for this problem.

One of the assumptions of this approach is that each instrument weakly shifts individuals into a specific treatment. In other words, Z_{1i} does not shift people between $T=0$ and $T=2$, and Z_{2i} does not shift people between $T=0$ and $T=1$. Are these assumptions satisfied with our data?

Yes, the monotonicity assumptions are satisfied. Increasing Z_1 raises only U_{1i}, leaving U_{2i} unchanged. Thus, turning on Z_1 only can move individuals into T_1. Similarly, increasing Z_2 raises only U_{2i}, leaving U_{1i} unchanged, so turning on Z_2 only can move individuals into T_2.

Part (d)

Show the following:

$$\frac{E(YT_j \mid Z_1=1) - E(YT_j \mid Z_1=0)}{E(T_j \mid Z_1=1) - E(T_j \mid Z_1=0)} = E(Y_{ji} \mid j\text{-1 complier}), \quad \text{for } j \in \{0,2\}$$

$$\frac{E(YT_j \mid Z_2=1) - E(YT_j \mid Z_2=0)}{E(T_j \mid Z_2=1) - E(T_j \mid Z_2=0)} = E(Y_{ji} \mid j\text{-2 complier}), \quad \text{for } j \in \{0,1\}$$

and:

$$\frac{E(YT_1 \mid Z_1=1) - E(YT_1 \mid Z_1=0)}{E(T_1 \mid Z_1=1) - E(T_1 \mid Z_1=0)} = E(Y_{1i} \mid \text{0-1 complier})\frac{\Pr(\text{0-1 complier})}{\Pr(\text{0-1 complier}) + \Pr(\text{2-1 complier})} + E(Y_{1i} \mid \text{2-1 complier})\frac{\Pr(\text{2-1 complier})}{\Pr(\text{0-1 complier}) + \Pr(\text{2-1 complier})}$$

$$\frac{E(YT_2 \mid Z_2=1) - E(YT_2 \mid Z_2=0)}{E(T_2 \mid Z_2=1) - E(T_2 \mid Z_2=0)} = E(Y_{2i} \mid \text{0-2 complier})\frac{\Pr(\text{0-2 complier})}{\Pr(\text{0-2 complier}) + \Pr(\text{1-2 complier})} + E(Y_{2i} \mid \text{1-2 complier})\frac{\Pr(\text{1-2 complier})}{\Pr(\text{0-2 complier}) + \Pr(\text{1-2 complier})}$$

Define for $j = 0, 1, 2$, $D_j(z_1, z_2) = \mathbb{I}\{T(z_1, z_2) = j\}$, so we can write observed outcomes as $Y_i = Y_{i0}D_{i0} + Y_{i1}D_{i1} + Y_{i2}D_{i2}$ and $Y_iT_{ji} = Y_{ij}D_{ji}$. Denote the complier groups

$$C_{01} = \{T(0, Z_2) = 0,\ T(1, Z_2) = 1\} \qquad \text{(0-1 complier)},$$

$$C_{21} = \{T(0, Z_2) = 2,\ T(1, Z_2) = 1\} \qquad \text{(2-1 complier)},$$
$$C_{02} = \{T(Z_1, 0) = 0,\ T(Z_1, 1) = 2\} \qquad \text{(0-2 complier)},$$
$$C_{12} = \{T(Z_1, 0) = 1,\ T(Z_1, 1) = 2\} \qquad \text{(1-2 complier)}.$$

For when Z_1 changes from 0 to 1, by monotonicity, we only consider the nontrivial transitions from $0-1$ and $2-1$ compliers. Thus,

$$D_0(1, Z_2) - D_0(0, Z_2) = -\mathbb{I}\{C_{01}\},$$

$$D_2(1, Z_2) - D_2(0, Z_2) = -\mathbb{I}\{C_{21}\}.$$

Consider $j = 0$. Then

$$\begin{aligned} E(YT_0 \mid Z_1 = 1) - E(YT_0 \mid Z_1 = 0) &= E\left[Y_0\{D_0(1, Z_2) - D_0(0, Z_2)\}\right] \\ &= -E\left[Y_0\mathbb{I}\{C_{01}\}\right] \\ &= -E(Y_0 \mid C_{01}) \Pr(C_{01}). \end{aligned}$$

Similarly,

$$E(T_0 \mid Z_1 = 1) - E(T_0 \mid Z_1 = 0) = -\Pr(C_{01}).$$

Thus,

$$\frac{E(YT_0 \mid Z_1 = 1) - E(YT_0 \mid Z_1 = 0)}{E(T_0 \mid Z_1 = 1) - E(T_0 \mid Z_1 = 0)} = E(Y_0 \mid C_{01}).$$

For $j = 2$, the same argument gives

$$\begin{aligned} E(YT_2 \mid Z_1 = 1) - E(YT_2 \mid Z_1 = 0) &= E\left[Y_2\{D_2(1, Z_2) - D_2(0, Z_2)\}\right] \\ &= -E\left[Y_2\mathbb{I}\{C_{21}\}\right] \\ &= -E(Y_2 \mid C_{21}) \Pr(C_{21}), \end{aligned}$$

and

$$E(T_2 \mid Z_1 = 1) - E(T_2 \mid Z_1 = 0) = -\Pr(C_{21}).$$

Thus,

$$\frac{E(YT_2 \mid Z_1 = 1) - E(YT_2 \mid Z_1 = 0)}{E(T_2 \mid Z_1 = 1) - E(T_2 \mid Z_1 = 0)} = E(Y_2 \mid C_{21}),$$

which proves the first identity.

For changes in Z_2 from 0 to 1, it shifts individuals only into T_2, so the only nontrivial transitions are in $0-2$ and $1-2$ compliers. Thus,

$$D_0(Z_1, 1) - D_0(Z_1, 0) = -\mathbb{I}\{C_{02}\},$$

$$D_1(Z_1, 1) - D_1(Z_1, 0) = -\mathbb{I}\{C_{12}\}.$$

For $j = 0$,

$$\begin{aligned} E(YT_0 \mid Z_2 = 1) - E(YT_0 \mid Z_2 = 0) &= E\left[Y_0\{D_0(Z_1, 1) - D_0(Z_1, 0)\}\right] \\ &= -E\left[Y_0 \mathbb{I}\{C_{02}\}\right] \\ &= -E(Y_0 \mid C_{02}) \Pr(C_{02}), \end{aligned}$$

and

$$E(T_0 \mid Z_2 = 1) - E(T_0 \mid Z_2 = 0) = -\Pr(C_{02}).$$

Thus,

$$\frac{E(YT_0 \mid Z_2 = 1) - E(YT_0 \mid Z_2 = 0)}{E(T_0 \mid Z_2 = 1) - E(T_0 \mid Z_2 = 0)} = E(Y_0 \mid C_{02}).$$

For $j = 1$,

$$\begin{aligned} E(YT_1 \mid Z_2 = 1) - E(YT_1 \mid Z_2 = 0) &= E\left[Y_1\{D_1(Z_1, 1) - D_1(Z_1, 0)\}\right] \\ &= -E\left[Y_1 \mathbb{I}\{C_{12}\}\right] \\ &= -E(Y_1 \mid C_{12}) \Pr(C_{12}), \end{aligned}$$

and

$$E(T_1 \mid Z_2 = 1) - E(T_1 \mid Z_2 = 0) = -\Pr(C_{12}).$$

Thus,

$$\frac{E(YT_1 \mid Z_2 = 1) - E(YT_1 \mid Z_2 = 0)}{E(T_1 \mid Z_2 = 1) - E(T_1 \mid Z_2 = 0)} = E(Y_1 \mid C_{12}).$$

We now show the two mixture formulas. For T_1, turning Z_1 on causes T_1 to gain individuals from both $0-1$ and $2-1$ compliers, i.e.

$$D_1(1, Z_2) - D_1(0, Z_2) = \mathbb{I}\{C_{01}\} + \mathbb{I}\{C_{21}\}.$$

Thus we see,

$$\begin{aligned} E(YT_1 \mid Z_1 = 1) - E(YT_1 \mid Z_1 = 0) &= E\left[Y_1\{D_1(1, Z_2) - D_1(0, Z_2)\}\right] \\ &= E\left[Y_1 \mathbb{I}\{C_{01}\}\right] + E\left[Y_1 \mathbb{I}\{C_{21}\}\right] \\ &= E(Y_1 \mid C_{01}) \Pr(C_{01}) + E(Y_1 \mid C_{21}) \Pr(C_{21}), \end{aligned}$$

and similarly,

$$E(T_1 \mid Z_1 = 1) - E(T_1 \mid Z_1 = 0) = \Pr(C_{01}) + \Pr(C_{21}).$$

Thus, we obtain the identity

$$\frac{E(YT_1 \mid Z_1 = 1) - E(YT_1 \mid Z_1 = 0)}{E(T_1 \mid Z_1 = 1) - E(T_1 \mid Z_1 = 0)} = E(Y_1 \mid C_{01}) \frac{\Pr(C_{01})}{\Pr(C_{01}) + \Pr(C_{21})} + E(Y_1 \mid C_{21}) \frac{\Pr(C_{21})}{\Pr(C_{01}) + \Pr(C_{21})}.$$

Similarly, T_2 gains both $0-2$ and $1-2$ compliers when Z_2 turns on, i.e.

$$D_2(Z_1, 1) - D_2(Z_1, 0) = \mathbb{I}\{C_{02}\} + \mathbb{I}\{C_{12}\}.$$

So likewise,

$$
\begin{aligned}
E(YT_2 \mid Z_2 = 1) - E(YT_2 \mid Z_2 = 0) &= E\left[Y_2\{D_2(Z_1, 1) - D_2(Z_1, 0)\}\right] \\
&= E\left[Y_2 \mathbb{I}\{C_{02}\}\right] + E\left[Y_2 \mathbb{I}\{C_{12}\}\right] \\
&= E(Y_2 \mid C_{02}) \Pr(C_{02}) + E(Y_2 \mid C_{12}) \Pr(C_{12}),
\end{aligned}
$$

and

$$E(T_2 \mid Z_2 = 1) - E(T_2 \mid Z_2 = 0) = \Pr(C_{02}) + \Pr(C_{12}),$$

so

$$\frac{E(YT_2 \mid Z_2 = 1) - E(YT_2 \mid Z_2 = 0)}{E(T_2 \mid Z_2 = 1) - E(T_2 \mid Z_2 = 0)} = E(Y_2 \mid C_{02}) \frac{\Pr(C_{02})}{\Pr(C_{02}) + \Pr(C_{12})} + E(Y_2 \mid C_{12}) \frac{\Pr(C_{12})}{\Pr(C_{02}) + \Pr(C_{12})},$$

proving the latter 2 identities.

Part (e)

Let's now make the following assumption (analogous to "comparable compliers" in his continuous setting):

$$E(Y_{1i} \mid \text{1-2 complier}) = E(Y_{1i} \mid \text{2-1 complier})$$

$$E(Y_{2i} \mid \text{1-2 complier}) = E(Y_{2i} \mid \text{2-1 complier})$$

Show how you can use the above assumption and what you proved in the previous part to identify

$$E(Y_{1i} - Y_{0i} \mid \text{0-1 complier})$$

and

$$E(Y_{2i} - Y_{0i} \mid \text{0-2 complier}).$$

Define the complier shares

$$\pi_{01} = \Pr(C_{01}), \quad \pi_{21} = \Pr(C_{21}), \quad \pi_{02} = \Pr(C_{02}), \quad \pi_{12} = \Pr(C_{12}),$$

which can be identified from the observed data via the identities from part (d):

$$\pi_{01} = E(T_0 \mid Z_1 = 0) - E(T_0 \mid Z_1 = 1),$$

$$\pi_{21} = E(T_2 \mid Z_1 = 0) - E(T_2 \mid Z_1 = 1),$$

$$\pi_{02} = E(T_0 \mid Z_2 = 0) - E(T_0 \mid Z_2 = 1),$$

$$\pi_{12} = E(T_1 \mid Z_2 = 0) - E(T_1 \mid Z_2 = 1).$$

For notational purposes, define the following ratios and mixtures from part (d):

$$R_0^{Z_1} = \frac{E(YT_0 \mid Z_1 = 1) - E(YT_0 \mid Z_1 = 0)}{E(T_0 \mid Z_1 = 1) - E(T_0 \mid Z_1 = 0)} = E(Y_0 \mid C_{01}),$$

$$R_2^{Z_1} = \frac{E(YT_2 \mid Z_1 = 1) - E(YT_2 \mid Z_1 = 0)}{E(T_2 \mid Z_1 = 1) - E(T_2 \mid Z_1 = 0)} = E(Y_2 \mid C_{21}),$$

$$R_0^{Z_2} = \frac{E(YT_0 \mid Z_2 = 1) - E(YT_0 \mid Z_2 = 0)}{E(T_0 \mid Z_2 = 1) - E(T_0 \mid Z_2 = 0)} = E(Y_0 \mid C_{02}),$$

$$R_1^{Z_2} = \frac{E(YT_1 \mid Z_2 = 1) - E(YT_1 \mid Z_2 = 0)}{E(T_1 \mid Z_2 = 1) - E(T_1 \mid Z_2 = 0)} = E(Y_1 \mid C_{12}),$$

$$M_1^{Z_1} = \frac{E(YT_1 \mid Z_1 = 1) - E(YT_1 \mid Z_1 = 0)}{E(T_1 \mid Z_1 = 1) - E(T_1 \mid Z_1 = 0),}$$

$$M_2^{Z_2} = \frac{E(YT_2 \mid Z_2 = 1) - E(YT_2 \mid Z_2 = 0)}{E(T_2 \mid Z_2 = 1) - E(T_2 \mid Z_2 = 0)}.$$

We have that

$$M_1^{Z_1} = E(Y_1 \mid C_{01})\frac{\pi_{01}}{\pi_{01} + \pi_{21}} + E(Y_1 \mid C_{21})\frac{\pi_{21}}{\pi_{01} + \pi_{21}},$$

and by the comparable compliers assumption,

$$M_1^{Z_1} = E(Y_1 \mid C_{01})\frac{\pi_{01}}{\pi_{01} + \pi_{21}} + R_1^{Z_2}\frac{\pi_{21}}{\pi_{01} + \pi_{21}}.$$

Solving for $E(Y_1 \mid C_{01})$, we get

$$E(Y_1 \mid C_{01}) = \frac{M_1^{Z_1}(\pi_{01} + \pi_{21}) - R_1^{Z_2}\pi_{21}}{\pi_{01}}.$$

Finally, using $R_0^{Z_1} = E(Y_0 \mid C_{01})$, we obtain the $0 - 1$ complier LATE

$$E(Y_1 - Y_0 \mid C_{01}) = \frac{M_1^{Z_1}(\pi_{01} + \pi_{21}) - R_1^{Z_2}\pi_{21}}{\pi_{01}} - R_0^{Z_1}.$$

We also have

$$M_2^{Z_2} = E(Y_2 \mid C_{02})\frac{\pi_{02}}{\pi_{02} + \pi_{12}} + E(Y_2 \mid C_{12})\frac{\pi_{12}}{\pi_{02} + \pi_{12}},$$

and by comparable compliers,

$$M_2^{Z_2} = E(Y_2 \mid C_{02})\frac{\pi_{02}}{\pi_{02} + \pi_{12}} + R_2^{Z_1}\frac{\pi_{12}}{\pi_{02} + \pi_{12}}.$$

Solving for $E(Y_2 \mid C_{02})$, we get

$$E(Y_2 \mid C_{02}) = \frac{M_2^{Z_2}(\pi_{02} + \pi_{12}) - R_2^{Z_1}\pi_{12}}{\pi_{02}}.$$

Finally, using $R_0^{Z_2} = E(Y_0 \mid C_{02})$, we obtain the $0 - 2$ complier LATE

$$E(Y_2 - Y_0 \mid C_{02}) = \frac{M_2^{Z_2}(\pi_{02} + \pi_{12}) - R_2^{Z_1}\pi_{12}}{\pi_{02}} - R_0^{Z_2}.$$

Since all ratios/mixtures/proportions are identified from observed data, the two counterfactual-specific treatment effects are identified.

Part (f)

Estimate

$$E(Y_{1i} - Y_{0i} \mid \text{0-1 complier})$$

and

$$E(Y_{2i} - Y_{0i} \mid \text{0-2 complier}).$$

How do they compare to what you estimated in part 1?

Using the observed data and the formulas for part (e), we estimate

$$E(Y_1 - Y_0|C_{01}) = 0.117, \quad E(Y_2 - Y_0|C_{02}) = 0.952,$$

which more closely match those estimated in part (a). They are still slightly off due to the comparable compliers assumption necessary to identify the true LATEs.

Part (g)

Suppose the data-generating process was slightly different. Draw a random variable

$$\pi = \begin{cases} 1 & \text{with probability } 0.5 \\ 2 & \text{with probability } 0.5 \end{cases}$$

And treatment is now:

$$T_{1i} = \mathbf{1}\{\pi = 1\} \times \mathbf{1}\{U_{1i} \geq 0\}$$
$$T_{2i} = \mathbf{1}\{\pi = 2\} \times \mathbf{1}\{U_{2i} \geq 0\}$$
$$T_{0i} = \mathbf{1}\{T_{1i} = 0 \ \& \ T_{2i} = 0\}$$

Redo Parts (a) and (b) with this new data-generating process. Has the conclusion changed? Why?

Hint: This is related to Proposition 2 in Kirkeboen, Leuven, and Mogstad (2016).

In this DGP, there are no longer cross-treatment margins, i.e. there are no $2 - 1$ or $1 - 2$ compliers. Thus, the standard 2SLS coefficients no longer mix these groups and the coefficient on T_1 is determined only by the $0 - 1$ compliers, and the coefficient on T_2 is determined only by the $0 - 2$ compliers. Indeed, in computing the estimates in Table 3, we find that the estimates are in fact very close.

Parameter	**Estimate**
$\hat{\beta}_1^{2SLS}$	0.491
$\hat{\beta}_2^{2SLS}$	1.004
$E(Y_1 - Y_0 \mid C_{01})$	0.500
$E(Y_2 - Y_0 \mid C_{02})$	1.000

Table 3: 2SLS Estimates and Direct Complier Treatment Effects